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ON NEGATIVE LIMIT SETS FOR ONE-DIMENSIONAL DYNAMICS

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ABSTRACT. In this paper we study the structure of negative limit sets of maps on the unit interval. We prove that every α-limit set is an ω-limit set, while the converse is not true in general. Surprisingly, it may happen that the space of all α-limit set of interval map is not closed in the Hausdorff metric (thus some ω-limit sets are never obtained as α-limit sets). Moreover, we prove the set of all recurrent points is closed if and only if the space of all α-limit sets is closed.

1. INTRODUCTION

Positive limit sets, so-called ω-limit sets, of the maps of the interval were deeply studied by many authors. For example [1] (see also [5]) shows that a nonempty subset M of the unit interval can be an ω-limit set if it is a union of intervals or a nowhere dense set. In this context it is also known that the space of all ω-limit sets is closed in the Hausdorff metric (see [4]) and that each ω-limit set is contained in the maximal one (see [4] or [15]). Recently these results were extended onto other classes of one-dimensional spaces, e.g. circle [12] or topological graphs [10].

While for homeomorphisms negative limit sets (called α-limit sets in the present paper) can be defined exactly in the same way as ω-limit sets, for non invertible maps it is not so obvious. One possibility is to take as an α-limit set the set of all accumulation points of the sequence $f^{-n}(\{x\})$. For example this approach is represented in [6]. There is also another possibility. Instead of looking at all possible preimages we can simply pick one negative trajectory and check accumulation points of this sequence. Of course obtained set will be usually smaller than the one obtained in the first approach, however it seems to better mimic the situation for homeomorphism. Therefore we adopt this notation in the paper. It is noteworthy that the idea of tracking a single negative trajectory can be generalized even more like in [8] where special α-limit sets were defined.

While α-limit sets seem to be very similar to ω-limit sets, they were not much studied so far. The reason for this can be twofold. First, it is much harder to deal with them, mainly because there are multiple choices for point in a negative trajectory. Secondly, images of open sets are usually not open under iteration of non-invertible map, so some tools like Baire Category Theorem can be harder to apply.

As we mentioned before, ω-limit sets are well characterized for interval maps, especially if the entropy of map is zero (e.g. see [5] and the references therein). In

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this paper we ask if similar results are true for α-limit sets, that is: must the set of all α-limit sets be closed, or can every α-limit set be characterized similarly to [5]? As we show, it is possible to some extent, but the characterization is not exactly the same.

We show that on the unit interval any α-limit set with respect to a negative (i.e backward) trajectory is locally expanding, and hence an ω-limit set. In the case of zero entropy maps we prove that any infinite α-limit set is perfect. Moreover, the set of all recurrent points is closed if and only if the space of all α-limit sets is closed, while there are also examples of maps on the interval with non-closed set of α-limit sets.

2. Preliminaries

In this paper, f always stands for a continuous map $f\colon X \to X$ acting on a compact metric space (X, d), denoted $f \in C(X)$ and in most cases $X = [0, 1]$. Often, we will additionally stress that fact in the assumptions of theorems.

Recall that a point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n > 0$ and *fixed* if $f(x) = x$. Denote by $\mathrm{Per}(f)$ the set of all periodic points of the map f. We will say that x is an accumulation point of a sequence $\langle z_n\rangle_{n=0}^{\infty}$ if there is a subsequence of $\langle z_n\rangle_{n=0}^{\infty}$ with the limit x.

Let x be a point in X. We define the *positive orbit* of x, denoted by $\mathrm{Orb}^+(x, f)$ as the set $\{f^n(x) : n = 0, 1, \ldots\}$. We will also sometimes identify $\mathrm{Orb}^+(x, f)$ with the sequence $\langle f^n(x)\rangle_{n=0}^{\infty}$ Similarly, given a nonempty set $A \subset X$ we denote $\mathrm{Orb}^+(A, f) = \bigcup_{n=0}^{\infty} f^n(A) = \bigcup_{x\in A} \mathrm{Orb}^+(x, f)$. The *$\omega$-limit set* of x is the set $\omega(x, f)$ of all limit points of the positive orbit of x regarded as a sequence. Recall that for every dynamical system $f \in C(X)$ we have

$$\omega(x, f) = \bigcup_{j=0}^{n-1} \omega(f^j(x), f^n) \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

and that a point x is said to be *recurrent* if $x \in \omega(x, f)$. We denote the set of all recurrent points by $\mathrm{Rec}(f)$.

If $f(A) \subset A$ then we say that the set A is *invariant* for f. If a stronger condition $f(A) = A$ is satisfied then we say that A is *strongly invariant*. A map f is *minimal*, if it has no proper closed invariant set, that is, if $K \subset X$ is nonempty, closed and $f(K) \subset K$ then $K = X$. A point is said to be *uniformly recurrent* if it is a member of a minimal set. It is not hard to verify that every uniformly recurrent point is recurrent.

A sequence $\langle x_{-n}\rangle_{n=0}^{\infty}$ of points in X is called a *negative orbit* or *negative trajectory through* x if $x_0 = x$ and $f(x_{-n-1}) = x_{-n}$ for every integer $n \geq 0$. If f is surjection then through every point there is at least one negative trajectory. Note that in general case, the negative orbit through x may not exists and, even if it exists, it may not be unique.

Definition 1. The *α-limit set* of a negative orbit $\langle x_{-n}\rangle_{n=0}^{\infty}$ is the set $\alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f)$ of all limit points of the sequence $\langle x_{-n}\rangle_{n=0}^{\infty}$.

Obviously, in contrast to the case of ω-limit set, the α-limit set does not depend only on the starting point x_0, but also the choice of a negative trajectory.

By $B(x, r)$ we will denote the open ball centered at x and with the radius r. We always endow $[0, 1]$ with the Euclidean metric, i.e. $d(x, y) = |x - y|$. We denote

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by $2^{[0,1]}$ the space of all nonempty closed subsets of $[0,1]$. The Hausdorff metric induced by d on the space $2^{[0,1]}$ is denoted by H_d. It is well known that $(2^{[0,1]}, H_d)$ is a compact metric space.

We denote by $\mathscr{P}(f), \mathscr{N}(f) \subset 2^{[0,1]}$ the sets of all ω-limit and α-limit sets of f, respectively, that is:

$$\begin{aligned} \mathscr{P}(f) &= \{\omega(x,f) : x \in [0,1]\}, \\ \mathscr{N}(f) &= \{\alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f) : \langle x_{-n}\rangle_{n=0}^{\infty} \subset [0,1] \text{ is a negative trajectory}\}. \end{aligned}$$

The following fact can be proved similarly as analogous property of ω-limit sets.

Lemma 2. *For any compact space (X,d), any $f \in C(X)$ and any negative trajectory $\langle x_{-n}\rangle_{n=0}^{\infty}$, the set $\alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f)$ is nonempty, closed and strongly invariant.*

Recall that a map $f \in C(X)$ is *transitive* if for any pair U and V of non-empty open subsets of X there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$. The proof of the next lemma is very simple, thus left to the reader. Observe that its immediate consequence is that every minimal set is also an α-limit set.

Lemma 3. *Let (X,d) be a compact metric space and assume that $f \in C(X)$ is transitive. Then X is an α-limit set of some negative trajectory $\langle z_{-n}\rangle_{n=0}^{\infty} \subset X$.*

Fix a map $f \in C([0,1])$. If $J \subset [0,1]$ is an interval, then we say that: J is *periodic* provided that $f^n(J) = J$ for some $n > 0$; *wandering* if $f^i(J) \cap f^j(J) = \emptyset$ provided that $i > j \geq 0$. If J is periodic interval, then its *period* is the minimal number $m > 0$ such that $f^m(J) = J$.

Recall that if f has no periodic points of least period not a power of 2, then by well known Sharkovsky's theorem [13] either f has periodic points with least period of finitely many different orders, say $1, 2, \ldots, 2^k$ for some k, or f has a periodic point of least period 2^n for every $n \geq 0$. In the second case f is referred to as a map of type 2^∞. It can be also proved (see [2] and [11]) that an interval map has positive topological entropy if and only if it has a periodic point with least period not a power of two. Thus the fact that f is of type at most 2^∞ in the Sharkovsky's ordering can be equivalently denoted as $h_{\text{top}}(f) = 0$. The notion of topological entropy can be found in any modern textbook on discrete dynamical systems, so we do not recall it.

3. Every α-limit set is ω-limit set on interval

First, we have to recall some notation and basic facts concerning ω-limit sets of interval maps. We adopt the terminology introduced in [4].

From this point on we fix a map $f \in C([0,1])$. Recall that a set V is *right* (resp. *left*) *unilateral neighborhood* of $x \in [0,1]$ if there exists an $\varepsilon > 0$ such that $[x, x+\varepsilon) \subset V$ (resp. $(x-\varepsilon, x] \subset V$). If T is a side of x (i.e. T means "right" or "left") then we can speak about *T-unilateral neighborhoods* of x.

Let $U \subset [0,1]$ be the union of finitely many pairwise disjoint compact and non-degenerate intervals and let $K \subset U$. Then by $f_U(K)$ we denote the set $f(K) \cap U$ and define recursively $f_U^n(K) = f_U(f_U^{n-1}(K))$, e.g. $f_U^2(K) = f(f(K) \cap U) \cap U$. By $\tilde{K}_U$ we denote the set $\bigcup_{i=1}^{\infty} f_U^i(K)$. When the set U is clear from the context, we will simply write $\tilde{K}$ instead of $\tilde{K}_U$.

Let $A \subset [0,1]$ be a closed set and let $x \in A$. We say that a side T of x is *A-covering* if for any union of finitely many closed intervals U such that $A \subset \operatorname{int} U$ and

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any closed T-unilateral neighborhood V of x there are finitely many components of $\tilde{V}_U$ such that the closure of their union covers A. If every $x \in A$ has A-covering side we call the set A *locally expanding* (with respect to f).

We will need the following two important results, proved first in [4].

Lemma 4 ([4, Theorem 2.12])**.** *Let $f \in C([0,1])$. A closed set A is an ω-limit set of f if and only if A is locally expanding.*

Lemma 5 ([4, Lemma 2.3])**.** *Let $K \subset U$ be an interval. Then $\tilde{K}$ is the union of two disjoint sets $\mathcal{A}, \mathcal{B}$ where:*

(1) $\mathcal{A}$ is a finite union of disjoint intervals and
(2) $\mathcal{B}$ is the union of orbits of finitely many pairwise disjoint wandering intervals.

Moreover, if K is closed then so are all of the wandering intervals defining $\mathcal{B}$.

Theorem 6. *For any $f \in C([0,1])$ and any negative trajectory $\langle x_{-n}\rangle_{n=0}^{\infty}$, the set $\alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f)$ is locally expanding.*

Proof. Denote $A = \alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f)$. Let $U = U_1 \cup \cdots \cup U_n$ be a set which is the union of the closed pairwise disjoint intervals such that $A \subset \operatorname{int} U$.

First let us assume that $\operatorname{int} A \neq \emptyset$. Then there is a compact interval $L \subset A$. There are $m > n$ such that $x_{-m}, x_{-n} \in L$. But then $x_{-n} \in f^{m-n}(L)$, thus $f^{m-n}(L) \cap L \neq \emptyset$ and consequently $f^{(k+1)(m-n)} \cap f^{k(m-n)}(L) \neq \emptyset$. This shows that the set $\bigcup_{i=0}^{\infty} f^{i(m-n)}(L)$ is connected, thus the set $W = \bigcup_{i=0}^{\infty} f^{i}(L)$ consists of finitely many intervals. By Lemma 2, $W \subset A$ and hence $\overline{W} \subset A$. We claim that $\overline{W} = A$. So let $y \in A \setminus \overline{W}$. Then there is a neighborhood V of y disjoint with $\overline{W}$ and such that $x_{-k} \in V$ for some $k > 0$. Since $L \subset A$ there is an $l > k$ such that $x_{-l} \in L$. But then $f^{l-k}(x_{-l}) = x_{-k}$ and hence $V \cap W \neq \emptyset$ which is a contradiction. Indeed $\overline{W} = A$. Now, if we fix any $y \in A$ then there is a side T of y such that any T-unilateral neighborhood V of y satisfies $\operatorname{int} V \cap W \neq \emptyset$. But if we fix such a neighborhood V then there is an interval $J \subset V \cap W$. But $f_U(J) = f(J) \cap U = f(J)$ and recursively $f_U^i(J) = f^i(J)$. Now, we can repeat arguments used previously for L obtaining that $\overline{Q} = A$ and Q is an union of finitely many intervals where

$$Q = \bigcup_{i=0}^{\infty} f^i(J) = \bigcup_{i=0}^{\infty} f_U^i(J) \subset \bigcup_{i=0}^{\infty} f_U^i(V).$$

We have just proved that y has a side T which is A-covering. Since y was arbitrarily, A is locally expanding.

Now consider the second case, that is, assume that A is a closed nowhere dense set. Let $y \in A$ and $\lim_{j\to\infty} x_{-k_j} = y$ from one side (i.e. x_{-k_j} is monotone), call this side T. We claim that T is an A-covering side of y. Let $V \subset U$ be a closed T-unilateral neighborhood of y. Without loss of generality we may assume that $x_{-k_j} \in V$ for all j and $\langle x_{-n}\rangle_{n=0}^{\infty} \subset U$, therefore $\langle x_{-n}\rangle_{n=0}^{\infty} \subset \tilde{V}$. By Lemma 5 the set $\tilde{V}$ can be presented as the union of two disjoint sets $\mathcal{A}$, $\mathcal{B}$ where $\mathcal{A}$ is the union of a finite family of disjoint intervals and $\mathcal{B}$ is the union of forward orbits of closed wandering intervals, say $J_1, \ldots, J_k$. Note that if $x_{-s} \in f^t(J_i)$ then $x_{-j} \notin f^t(J_i)$ for any $j \geq s$. Thus there is N such that $x_{-N-j} \notin \bigcup_{i=1}^{k} \operatorname{Orb}^+(J_i)$ for any $j \geq 0$. But if we remove first N elements of the sequence $\langle x_{-n}\rangle_{n=0}^{\infty}$ then it does not affect α-limit set defined by this sequence, thus without loss of generality we may assume

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that $\langle x_{-j}\rangle_{j=0}^{n} \subset \mathcal{A}$. Therefore $A = \alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f) \subset \overline{\mathcal{A}}$ whence T is indeed an A-covering side of y and thus A is locally expanding since y was arbitrary. □

Corollary 7. *Let $f \in C([0,1])$. Then any α-limit set $\alpha(\langle x_{-n}\rangle_{n=0}^{\infty}, f)$ is an ω-limit set of f.*

Proof. It is enough to apply Lemma 4 to the statement of Theorem 6. □

Remark 8. *It is easy to see that Theorem 6 cannot be true in general setting of compact metric space (even one-dimensional). For example, if we take the unit circle S^1 made of fixed points and infinite line S unwinding from S^1 to the fixed point in the center, then putting slow forward motion on S we can make the map continuous. It is also evident that S^1 is α-limit set of (the unique) negative trajectory of any point on S while it cannot be ω-limit set.*

4. Maps of type 2^∞

If A is an infinite ω-limit set of an interval map f such that $A \cap \operatorname{Per}(f) = \emptyset$ then we say it is a *solenoid*. It was first proved by Sharkovsky [14] (see also [7]) that if f has zero topological entropy then all infinite maximal ω-limit sets are solenoids. The following fact is [16, Theorem 3.5].

Lemma 9. *Let $f \in C([0,1])$ be a map with $h_{top}(f) = 0$ and let M be a maximal infinite ω-limit set. Then there is a sequence $\langle I_n\rangle_{n=0}^{\infty}$ of (not necessarily closed) periodic intervals such that for any n*

(1) I_n has period 2^n,
(2) $I_{n+1} \cup f^{2^n}(I_{n+1}) \subset I_n$,
(3) $\operatorname{Orb}(I_n) \supset \tilde{\omega}$,
(4) $M \cap f^i(I_n) \neq \emptyset$ for every i,

Let I_n be the sequence of periodic intervals provided by Lemma 9 for $\omega(x,f)$ and denote $I_n^i = f^i(I_n)$ for $i = 0, 1, \ldots, 2^n - 1$. In this setting the following fact can be proved (see Lemma 16 and 18 in Section VI of [3]):

Lemma 10. *In the setting of Lemma 9, for any nested sequence $I_1^{a_1} \supset I_2^{a_2} \supset \ldots$ put $K = \bigcap_{j=1}^{\infty} I_j^{a_j}$. Then either $K = \{y\}$ and $y \in \omega(x,f)$ or $K = [y,z]$ and $K \cap \omega(x,f) = \{y,z\}$. When $K = \{y\}$ then y is recurrent, and if $K = [y,z]$ then at most one of points y,z is not recurrent.*

If any of the points y,z above is recurrent then it is uniformly recurrent.

We will also need the following fact (see Proposition 7 in Section VI of [3]).

Lemma 11. *If $h_{top}(f) = 0$ then for any $x \in [0,1]$ the set $\omega(x,f)$ contains at most one minimal set.*

Theorem 12. *Let $f \in C([0,1])$ be a map with $h_{top}(f) = 0$ and let M be an infinite α-limit set of some negative trajectory. Then M is perfect.*

Proof. By Corollary 7 there is $x \in [0,1]$ such that $M = \omega(x,f)$. Let sequence $\langle I_n\rangle_{n=0}^{\infty}$ be provided for M by Lemma 9. On the contrary, assume that q is an isolated point of M. First note that q cannot be recurrent, because then it is a periodic point and so cannot belong to I_n for sufficiently large n. Let a_n be a sequence such that $q \in K = \bigcap_{j=1}^{\infty} I_j^{a_j}$. Then, by Lemma 10, K is an interval, q is one of its endpoints and the second endpoint is a recurrent point $p \in M$.

Nonlinear analysis: theory, methods & applications. 2012, vol. 75, issue 6, p. 3262-3267. http://dx.doi.org/10.1016/j.na.2011.12.030

Furthermore, there must exist a sequence of distinct isolated points of $q_{-n} \in M$ and non-isolated points $p_{-n} \in M$ such that $f^j(q_{-n}) = q_{-n+j}$, $f^j(p_{-n}) = p_{-n+j}$ and $q_0 = q$, $p_0 = p$. For simplicity of notation, assume that $p_{-n} < q_{-n}$ for every n (then we can simply write $[p_{-n}, q_{-n}]$ instead of convex hull of these points). Again, using Lemma 10 we see that the only possibility is that $f([p_{-n}, q_{-n}]) = [p_{-n+1}, q_{-n+1}]$, since $f([p_{-n}, q_{-n}]) = f(\bigcap_{j=1}^{\infty} I_j^{a_j}) \subset \bigcap_{j=1}^{\infty} I_j^{a_j+1}$ and $\bigcap_{j=1}^{\infty} I_j^{a_j+1}$ is a closed interval (degenerate or not) which can intersect M only at its endpoints. By similar argument as the above (and the fact that none of points p_{-n}, q_{-n} is periodic) we see that all the intervals $[p_{-n}, q_{-n}]$ are pairwise disjoint.

First we claim that $q \in \operatorname{int}\operatorname{Orb}^+(I_n, f)$ for every $n \in \mathbb{N}$. If not, then q is an endpoint of some $I_N^{a_N}$ and then it must be an endpoint of every $I_m^{a_m}$, $m > N$ since these intervals form a nested sequence. But M is infinite, therefore $\operatorname{Orb}^+(x, f) \cap \operatorname{int} I_N \neq \emptyset$ and so $f^k(x) \in \operatorname{Orb}^+(I_N, f)$ for all k sufficiently large, say $k > K$ for some $K > 0$. Then, there exists $k > K$ such that $f^k(x) \in (p, q)$. Now we have two possibilities, that is, either $f^j([p, q])$ reduces to a single point for some j or not. If it does, then $f^l(x) \in M$ for some $l > k$ and therefore sequence $f^j(x)$, $j > l$ consists of non-isolated points of M, in particular cannot have q as its accumulation point. In the second case, by arguments similar as for backward orbit of $[p, q]$, we see that $[p, q]$ is a wandering interval, and again $f^j(x)$ as a member of wandering interval can never return to $[p, q]$. Thus in both cases we have a contradiction.

By the assumptions $M = \alpha(\langle y_{-n}\rangle_{n=0}^{\infty}, f)$ for some negative orbit. Let $\varepsilon < |p-q|$ be such that $(q-\varepsilon, q+\varepsilon) \cap M = \emptyset$. Then $(q-\varepsilon, q+\varepsilon)$ intersects the unique set $I_n^{a_n}$ if n is sufficiently large. Next observe that since q is in the interior of $\operatorname{Orb}^+(I_n, f)$ for every n then $\langle y_{-n}\rangle_{n=0}^{\infty} \subset \bigcap_{n=1}^{\infty} \operatorname{Orb}^+(I_n, f)$. But by the choice of ε we get that there are infinitely many k such that $y_{-k} = q$ which is impossible, since q is not periodic. The proof is finished. □

It is well known that the set of all ω-limit sets of any continuous map on the unit interval is always closed (in the space $2^{[0,1]}$). Unfortunately, it is no longer the case if we consider α-limit sets. This fact is a consequence of construction in [17]. For completeness we present its proof in a more detail.

Theorem 13. *There exists a map $\chi \in C([0,1])$ such that:*

(1) there is a point $x \in [0,1]$ such that $\omega(x, \chi) \neq \alpha(\langle z_{-n}\rangle_{n=0}^{\infty}, \chi)$ for any backward trajectory of any point $z_0 \in [0,1]$, i.e. $\omega(x, \chi) \notin \mathscr{N}(\chi)$.

(2) The set $\mathscr{N}(\chi)$ is not closed in $2^{[0,1]}$, i.e. with respect to the Hausdorff metric.

Proof. First, let us recall in more detail a construction from [17], since it is core of the proof. In Proposition 2.1 of [17] there is constructed a map μ with of type 2^{∞} (therefore $h_{\mathrm{top}}(\mu) = 0$) and the unique maximal infinite ω-limit Q of μ which is a Cantor set and satisfies additional condition (see the proof of [17, Proposition 2.1(i)]):

(Q1): Periodic intervals I_n provided by Lemma 9 can be chosen in such a way that:

(i) Lebesgue measure of $\operatorname{Orb}^+(I_n, \mu)$ is bounded by $(2/5)^n$, in particular diameter of any interval $\mu^i(I_n)$ is bounded by $(2/5)^n$,

(ii) Endpoints of I_n are periodic orbits.

Nonlinear analysis: theory, methods & applications. 2012, vol. 75, issue 6, p. 3262-3267. http://dx.doi.org/10.1016/j.na.2011.12.030

Denote $a = \min(Q)$. Without loss of generality (replacing I_n by its image under μ if necessary) we may assume that $\{a\} = \bigcap_{n=1}^{\infty} I_n$. Then if we denote by p_n the left endpoint of I_n, by (i) and (ii) we have that $\lim_{n\to\infty} H_d(\mathrm{Orb}^+(p_n,\mu),Q) = 0$ and simply by the definition $p_n < a = \min(Q)$ for every n. The map μ has also the property that the preimage $\mu^{-1}(x)$ is a finite set for $x \in [0,1]$ (see [17, Proposition 2.1(v)]). Therefore the set $A = \mathrm{Orb}^+(a,\mu) \cup \bigcup \mu^{-n}(a)$ is countable, say $A = \{x_i : i \in \mathbb{Z}\}$ where $x_i \neq x_j$ for $i \neq j$. Furthermore, if $\mu^n(x_i) = x_j$ for some $n > 0$ then $i \neq j$ and $x_i \notin \mathrm{Orb}^+(x_j,\mu)$, as otherwise a would be an eventually periodic point. Note that just by the definition, both sets A and $[0,1] \setminus A$ are invariant, i.e. $T(A) = A$ and $\mu([0,1] \setminus A) = [0,1] \setminus A$ There is also a function $\phi\colon \mathbb{Z} \to \mathbb{Z}$ so that $\mu(x_i) = x_{\phi(i)}$.

Now we will perform Donjoy's type construction, replacing every point in A by an interval. Strictly speaking, remove all points x_i from $[0,1]$ and fill each obtained hole with an interval J_i of length $2^{-|i|}$. This way a new continuous map χ is defined on the extended space in such a manner that:

(a) each interval J_i is mapped homeomorphically onto $J_{\phi(i)}$,
(b) if all intervals J_i are collapsed back into single points then F reverts back to the map μ.

By the construction the domain of χ is isometric to $[0,4]$, so we can assume that $\chi\colon [0,4] \to [0,4]$. In this way every interval J_i is identified with $J_i = [a_i,b_i] \subset (0,4)$ and there is a quotient map $\pi : [0,4] \to [0,1]$ that does not increase distance, collapses every interval J_i into a single point (i.e. $\pi(J_i) = \{x_i\}$), and has the property that $\mu \circ \pi = \pi \circ \chi$. Since χ is at most two-to-one extension of μ, we have $h_{\mathrm{top}}(\mu) = h_{\mathrm{top}}(\chi) = 0$ (other way to see it, is that our construction does not produce new periodic points). Note that π is invertible on $[0,1] \setminus A$, so there is a sequence of periodic points $\langle q_n \rangle_{n=1}^{\infty}$ such that $\pi(q_n) = p_n$ and $q_n = \min \mathrm{Orb}^+(q_n,\chi)$ and if we denote $q = \lim_{n\to\infty} q_n$ then $\pi(q) = a$.

Now, if we denote by M the limit in the Hausdorff metric of a convergent subsequence of $\mathrm{Orb}^+(q_n,\chi)$ then $M = \omega(x,\chi)$ for some $x \in [0,4]$ since the space of all ω-limit sets is closed. By the definition we have $q = \min M$, thus q is isolated in M as an endpoint of some wandering interval J_n. Orbit of a is infinite, then also M is infinite. But M is not a perfect set, hence cannot be an α-limit set by Theorem 12. This proves both (1) and (2) at the same time. □

Theorem 14. *If $f \in C([0,1])$ has zero topological entropy, then M is an α-limit set for f if and only if M is a minimal set for f.*

Proof. Let $M = \alpha(\langle x_n \rangle_{n=0}^{\infty}, f)$ be an α-limit set for M. If M is finite then it is a periodic orbit by Lemma 2. If M is infinite then it is perfect by Theorem 12 and an ω-limit set by Theorem 6. By Lemma 10 there can be at most countably many non-recurrent points in M and every recurrent point is uniformly recurrent. Then there are no non-recurrent points in M, since by Lemma 11, M contains the unique minimal set S which contains all but at most countably points of M. Therefore $S = M$ and the proof is completed. □

Theorem 15. *Assume that $f \in C([0,1])$ has zero topological entropy. Then the following conditions are equivalent:*

(1) $\mathscr{N}(f)$ is a closed subset of the hyperspace $2^{[0,1]}$;

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(2) *if* $x \in [0,1]$*,* $M = \omega(x,f)$ *and* $a \in M$ *is isolated in* M*, then either* M *is finite or* $a \notin \overline{\mathrm{Per}(f)}$*;*

(3) $\mathrm{Rec}(f) = \overline{\mathrm{Rec}(f)}$*.*

Proof. First we prove implication (1) $\implies$ (2). Let us assume on the contrary that there is an infinite ω-limit set $\omega(x,f)$ with an isolated point $a \in \overline{\mathrm{Per}(f)}$. Let M be the maximal ω-limit set containing $\omega(x,f)$. By Lemmas 10 and 11 we see that a is isolated also in M. Since $a \in \overline{\mathrm{Per}(f)}$ there is a sequence $\langle p_n\rangle_{n=0}^{\infty} \in \mathrm{Per}(f)$ such that $\lim_{n\to\infty} p_n = a$. But $2^{[0,1]}$ is compact, so without loss of generality we may assume that the following limit $\lim_{n\to\infty} \mathrm{Orb}^+(p_n) = D$ exists in $2^{[0,1]}$. By [4] the set $\mathscr{P}(f)$ is closed and $\mathrm{Orb}^+(p_n) \in \mathscr{P}(f)$, so $D \in \mathscr{P}(f)$. Obviously $a \in D$, therefore $D \subset M$ and thus a is isolated in D too. But D cannot be periodic orbit, so it is not minimal set. Note that $D \notin \mathscr{N}(f)$ by Theorem 14, which is a contradiction.

Implication (2) $\implies$ (3) is a simple consequence of the well known fact that $\overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)}$ (e.g. see Proposition II.15 in [3]). Simply, if $a \in \overline{\mathrm{Rec}(f)} \setminus \mathrm{Rec}(f)$ then there is a sequence of periodic points p_n such that $\lim_{n\to\infty} p_n = a$. But each set $\mathrm{Orb}^+(p_n, f)$ is also an ω-limit set, and so is the set $D = \lim_{n\to\infty} \mathrm{Orb}^+(p_n, f)$, where the limit is taken in the Hausdorff metric (going to a subsequence, we may assume that D is well defined). Obviously $a \in D$, and a is not periodic as a nonrecurrent point. Then D is infinite and again by Lemmas 10 and 11 we see that a is isolated in D. This is in contradiction with (2), since we assumed that $a \in \overline{\mathrm{Rec}(f)} = \overline{\mathrm{Per}(f)}$.

For the proof of the last implication (3) $\implies$ (1) fix any convergent sequence $\langle M_n\rangle_{n=0}^{\infty} \subset \mathscr{N}(f)$, say $\lim_{n\to\infty} M_n = M$. Note that $M \in \mathscr{P}(f)$ by Corollary 7 and the fact that $\mathscr{P}(f)$ is closed. If M is finite then we are done, so assume that M is infinite. Fix any $x \in M$. Then there exists $z_n \in M_n$ such that $\lim_{n\to\infty} z_n = x$. Observe that $z_n \in \mathrm{Rec}(f)$ and so also $x \in \mathrm{Rec}(f) = \overline{\mathrm{Rec}(f)}$. This shows by Lemmas 10 and 11 that M is minimal and so $M \in \mathscr{N}(f)$ by Theorem 14. The proof is complete. □

Remark 16. *It may happen that some infinite (maximal)* ω*-limit sets of interval map with zero entropy have isolated points but* $\mathscr{N}(f)$ *is closed. Such example is provided, e.g. by* [9]*, since map constructed there satisfies condition* (3) *of Theorem 15.*

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